

UNIQUENESS OF THE EXTREME CASES IN THEOREMS OF DRISKO AND ERDŐS-GINZBURG-ZIV

RON AHARONI, DANI KOTLAR, AND RAN ZIV

ABSTRACT. Drisko [4] proved (essentially) that every family of $2n - 1$ matchings of size n in a bipartite graph possesses a partial rainbow matching of size n . In [5] this was generalized as follows: Any $\lfloor \frac{k+2}{k+1}n \rfloor - (k+1)$ matchings of size n in a bipartite graph have a rainbow matching of size $n - k$. We extend this latter result to matchings of not necessarily equal cardinalities. Settling a conjecture of Drisko, we characterize those families of $2n - 2$ matchings of size n in a bipartite graph that do not possess a rainbow matching of size n . Combining this with an idea of Alon [2], we re-prove a characterization of the extreme case in a well-known theorem of Erdős-Ginzburg-Ziv in additive number theory.

1. INTRODUCTION

Let $\mathcal{C} = (C_1, \dots, C_m)$ be a system of sets. A set that is the range of an injective partial choice function from \mathcal{C} is called a *rainbow set*, and is also said to be *multicolored* by \mathcal{C} . If ϕ is such a partial choice function and $C_i \in \text{dom}(\phi)$ we say that C_i *colors* $\phi(C_i)$ in the rainbow set. If the elements of C_i are sets then a rainbow set is said to be a (*partial*) *rainbow matching* if its range is a matching, namely it consists of disjoint sets.

Let A be an $m \times n$ matrix whose elements are symbols. A *partial transversal* in A is a set entries with no two in the same row or column, and no two sharing the same symbol. If the partial transversal is of size $\min(m, n)$ then it is called *full*, or simply a *transversal*. A famous conjecture of Ryser-Brualdi-Stein (see [8]) is that in an $n \times n$ Latin square there is a partial transversal of size $n - 1$. Drisko [4] showed that this is true if the assumption is “doubled”:

Theorem 1.1. *Let A be an $m \times n$ matrix in which the symbols in each row are all distinct. If $m \geq 2n - 1$, then A has a full transversal.*

In [1] this was formulated in a slightly more general version:

Theorem 1.2. *Any family $\mathcal{M} = (M_1, \dots, M_{2n-1})$ of matchings of size n in a bipartite graph possesses a rainbow matching of size n .*

Theorem 1.1 follows upon noting that in a matrix A as in the theorem every column defines a matching between rows and symbols. Theorem 1.2 is sharp, as shown by the following example:

Example 1.3. For $n > 1$ let M_1, \dots, M_{n-1} be all equal to the matching consisting of all even edges in a cycle of length $2n$, and M_n, \dots, M_{2n-2} be all equal to the matching consisting of all odd edges in this cycle. Together these are $2n - 2$ matchings of size n not having a rainbow matching of size n .

This example can be formulated in matrix form, which proves that also Theorem 1.1 is sharp. Drisko ([4], Conjecture 2) conjectured that this is the only extreme example. The main aim of this paper is to prove this conjecture, also in the wider context of matchings.

Theorem 1.4. *Let $\mathcal{M} = (M_1, \dots, M_{2n-2})$ be a family of matchings of size n in a bipartite graph. If \mathcal{M} does not possess a partial rainbow matching of size n , then $\bigcup \mathcal{M}$ is the edge set of a cycle of length $2n$, half of the members of \mathcal{M} are the matching consisting of the even edges of this cycle, and the other half are equal to the matching consisting of the odd edges in the cycle.*

The research of the first author was supported by an ISF grant, BSF grant no. 2006099 and by the Discount Bank Chair at the Technion.

Recently, Barát, Gyarfás and Sarkozy [5] proved a generalization of Theorem 1.2:

Theorem 1.5. Any $\lfloor \frac{k+2}{k+1}n \rfloor - (k+1)$ matchings of size n in a bipartite graph have a rainbow matching of size $n - k$.

In Section 2 we shall prove a still more general result:

Theorem 1.6. Let $a \leq m$, and let M_1, \dots, M_m be matchings ordered so that $|M_1| \leq |M_2| \leq \dots \leq |M_m|$.

Suppose that

$$(1) \quad \sum_{i \leq m-a+1} (|M_i| - a + 1) \geq a$$

Then there exists a rainbow matching of size a .

Note that Theorem 1.2 is the case $|M_i| = n$, $m = 2n - 1$.

2. MULTICOLORED PATHS

The proof of Theorem 1.4 is based on a simple fact (Theorem 2.1 below) on multicolored paths in networks. We shall use the following notation for paths in a directed graph. The vertex set of a path P is denoted by $V(P)$ and its edge set by $E(P)$. If \mathcal{P} is a set of paths, we write $E[\mathcal{P}]$ for $\bigcup_{P \in \mathcal{P}} E(P)$. If P is a path and $v \in V(P)$ we write vP for the part of P between v and the last vertex of P . Similarly, Pv is the part of P ending at v . If P, Q are paths, $u \in V(P)$ and $v \in V(Q)$ and uv is an edge, we write $PuvQ$ for the trail Pu concatenated with the edge uv and then with vQ (a *trail* is a path with repetition of vertices allowed). A path starting at a vertex a and ending at a vertex b is called an $a - b$ path.

A *network* \mathcal{N} is a triple (D, s, t) , where $D = (V, E)$ is a directed graph, and s (the “source”) and t (the “sink”) are two distinguished vertices in V . We assume that no edge goes into s and no edge leaves t . We write V° for $V \setminus \{s, t\}$. Given an $s - t$ path P , we write $V^\circ(P)$ for $V(P) \setminus \{s, t\}$. Two $s - t$ paths P and Q are said to be *innerly disjoint* if $V^\circ(P) \cap V^\circ(Q) = \emptyset$.

Let \mathcal{P} be a family (that is, a multiset) of $s - t$ paths in D , and suppose that $\mathcal{P} = \bigcup_{i \leq m} \mathcal{P}_i$, where each \mathcal{P}_i is a set of pairwise innerly disjoint paths. Let $\mathcal{L} = (\mathcal{P}_1, \dots, \mathcal{P}_m)$. A path Q in D is said to be \mathcal{L} -*multicolored* if its edge set is a rainbow set for the family $(E[\mathcal{P}_1], E[\mathcal{P}_2], \dots, E[\mathcal{P}_m])$, namely if each of its edges belongs to a path from a different \mathcal{P}_i . A vertex $v \in V$ is said to be *reachable* if there is an $s - v$ \mathcal{L} -multicolored path. Denote the set of reachable points by $R(\mathcal{L})$.

Theorem 2.1. $|R(\mathcal{L})| > |\mathcal{P}|$.

Proof. By induction on $|\mathcal{P}|$. Since $R(\emptyset) = \{s\}$, the theorem is true for $|\mathcal{P}| = 0$. Suppose that the result is true for any network and all families of paths with fewer than $|\mathcal{P}|$ paths. Let X be the set of all vertices x such that $sx \in E(P)$ for some $P \in \mathcal{P}_1$. Contract s and X to a single vertex s' (so, if there are r paths in \mathcal{P}_1 , $r + 1$ vertices are contracted to the single vertex s'). For each $P \in \mathcal{P}$ let v be the last vertex on P belonging to the set $\{s\} \cup X$, and let $P' = vP$. Then P' is an $s' - t$ path. Let $\mathcal{P}'_i = \{P' \mid P \in \mathcal{P}_i\}$, and let $\mathcal{L}' = (\mathcal{P}'_1, \dots, \mathcal{P}'_m)$. Clearly, the paths in each \mathcal{P}'_i are pairwise innerly disjoint. By the induction hypothesis $|R(\mathcal{L}')| > \sum_{2 \leq j \leq m} |\mathcal{P}_j|$. We claim that $R(\mathcal{L}') \setminus \{s'\} \subseteq R(\mathcal{L})$. To show this, let $y \in R(\mathcal{L}') \setminus \{s'\}$, and let Q be an \mathcal{L}' -multicolored $s' - y$ path. Let $s'z$ be the first edge on Q , and assume that it belongs to some path P' , where $P \in \mathcal{P}_i$ for some $i \geq 2$. By the definition of the contracted vertex s' , this means that either $sx \in E(P)$ or $xz \in E(P)$ for some $x \in X$. In the first case the $s - t$ path Q' (with s replacing s' on Q') is \mathcal{L} -multicolored (not using \mathcal{P}_1). In the second case the $s - t$ path sxQ' (with x replacing s' on Q') is \mathcal{L} -multicolored, with sx colored by \mathcal{P}_1 .

Since all vertices in X are \mathcal{L} -reachable, it follows that $|R(\mathcal{L})| \geq |R(\mathcal{L}')| + |X| > \sum_{2 \leq j \leq m} |\mathcal{P}_j| + |X| = |\mathcal{P}|$, completing the proof. \square

Corollary 2.2. If $|\mathcal{P}| > |V^\circ|$ then $t \in R(\mathcal{L})$.

Given a matching F , a path P is said to be F -*alternating* if one of each pair of adjacent edges in P belongs to F . An F -alternating path is called *augmenting* if its first and last vertices do not belong to $\bigcup F$. If A is

an augmenting F -alternating path then the symmetric difference of $E(A)$ and F is a matching larger than F by 1. Each connected component of the union of two matchings is an alternating path with respect to each of the matchings, and hence we have the following well known observation:

Lemma 2.3. *If G and H are matchings in a graph, and $|H| = |G| + q$, then $H \cup G$ contains at least q vertex disjoint augmenting G -alternating paths.*

Proof of Theorem 1.6 from Corollary 2.2. Let M_1, \dots, M_m be matchings in a bipartite graph with sides A, B . We shall show that if F is a rainbow matching of size $p < a$ then there exists a larger rainbow matching. Let J be the set of indices j such that M_j is not represented in F . Contract all vertices in $A \setminus \bigcup F$ to a single vertex s , and all vertices in $B \setminus \bigcup F$ to a single vertex t .

For each edge $f \in F$ let $v(f)$ be a vertex in \mathcal{N} . By Lemma 2.3 for each matching M_j not represented in F there exists a family \mathcal{P}_j of $\max(|M_j| - a + 1, 0)$ disjoint augmenting F -alternating paths. Let $\mathcal{L} = (\mathcal{P}_1, \dots, \mathcal{P}_{m-a+1})$.

Each path T in each \mathcal{P}_j gives rise to an $s-t$ path $P(T)$ in \mathcal{N} obtained by replacing every edge $ab \in E(T)$, where $a \in A \cap f$, $b \in B \cap g$ (here $f, g \in F$) by the edge $v(f)v(g)$ in \mathcal{N} , every edge $ab \in E(T)$ in which $a \notin \bigcup F$ and $b \in f \in F$ by the edge $sv(f)$, and every edge $ab \in E(T)$ in which $b \in B \setminus \bigcup F$ and $a \in f \cap A$, $f \in F$ by $v(f)t$. If T does not meet F at all, namely it consists of a single edge, then $P(T) = st$.

By the assumption (1) and by Corollary 2.2, there exists an $s-t$ \mathcal{L} -multicolored path, which translates to a multicolored augmenting F -alternating path L . Taking the symmetric difference of L and F yields a rainbow matching larger than F , as desired. \square

Theorem 1.5 is probably not sharp. In fact, it has been conjectured in [1] that n matchings of size n in a bipartite graph have a rainbow matching of size $n-1$.

3. THE EXTREME CASE IN COROLLARY 2.2

The aim of this section is to characterize the examples showing the tightness of Corollary 2.2, namely those sets \mathcal{P} of $|V| - 2$ paths that do not possess a \mathcal{P} -multicolored $s-t$ path.

Definition 3.1. A system \mathcal{P} of $s-t$ paths in a network \mathcal{N} with the notation above is called *regimented* if there exist pairwise innerly disjoint $s-t$ paths P_1, \dots, P_k and sets of paths $\mathcal{P}_1, \dots, \mathcal{P}_k$ such that $\mathcal{P} = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_k$, and each \mathcal{P}_j is a set of $|E(P_j)| - 1$ paths, all identical to P_j .

Clearly, a regimented set \mathcal{P} of $s-t$ paths does not have a \mathcal{P} -multicolored $s-t$ path.

Assertion 3.2. *If \mathcal{P} is regimented and $|\mathcal{P}| = |V^\circ|$, then $\bigcup \{V(P) \mid P \in \mathcal{P}\} = V$.*

Proof. This follows from the fact that if $\mathcal{P}_1, \dots, \mathcal{P}_k$ is a regimentation of \mathcal{P} , with corresponding paths P_i , then $|\mathcal{P}_i| = |V(P_i) \setminus \{s, t\}|$ for each $i \leq k$. \square

Theorem 3.3. *If $|\mathcal{P}| = |V^\circ|$ and \mathcal{P} is not regimented, then there exists an $s-t$ \mathcal{P} -multicolored path.*

To prove the theorem, let $m = |V^\circ|$. Assume, for contradiction, that \mathcal{P} does not have a multicolored $s-t$ path. We shall show, by induction on $|V|$, that \mathcal{P} is regimented. Assume that the result is true for networks with fewer vertices. Let x be a vertex such that $sx \in P_1$. If $x = t$ then the path $P_1 = st$ is multicolored, so we may assume that $x \neq t$. As in the proof above of Theorem 1.2, contract x and s , obtaining a network \mathcal{N}' . Like in that proof, for every $P = P_i$ we write $P' - P'_i$ for the $s'-t$ path obtained from P by this contraction. Let $\mathcal{P}' = \{P'_i \mid i = 2, \dots, m\}$. If there exists in \mathcal{P}' a \mathcal{P}' -multicolored $s'-t$ path, then as in the above proof it follows that there exists a \mathcal{P} -multicolored $s-t$ path. So, we may assume that there is no such path, which, by the induction hypothesis, implies that \mathcal{P}' is regimented. Let Q_1, \dots, Q_k be the paths regimenting \mathcal{P}' , and let $\mathcal{Q}_1, \dots, \mathcal{Q}_k$ be the corresponding sets of paths (so $Q = Q_j$ for every $Q \in \mathcal{Q}_j$).

Assertion 3.4. $\bigcup_{i=1}^k V^\circ(Q_i) = V^\circ \setminus \{x\}$.

Proof. If $|\bigcup_{i=1}^k V^\circ(Q_i)| < |V^\circ| - 1$ then $|V^\circ(\mathcal{P}')| < |\mathcal{P}'|$ and by Corollary 2.2 \mathcal{P}' has a multicolored path, contradicting our assumption. Hence, $|\bigcup_{i=1}^k V^\circ(Q_i)| = |V^\circ| - 1$ and the result follows by Assertion 3.2. \square

Assertion 3.5. *Let $k \geq 2$. If $uv \in E(P_k)$ then there exists i such that $\{u, v\} \subseteq V(Q_i)$.*

Proof. Assuming negation, by Assertion 3.2 there exist $i \neq j$ such that $u \in V(Q_i) \setminus V(Q_j)$ and $v \in V(Q_j) \setminus V(Q_i)$. $P \in \mathcal{P}$ be a path such $Q_i = P'$. Then the path $PuvQ_j$ is multicolored, where the edge sx , if present in P , is colored by P_1 , the edges in Q_i by paths from Q_i , the edge uv by P_k and the edges of Q_j by paths from Q_j . \square

We can now prove Theorem 3.3.

Case I: The path xP_1 , with x replaced by s' , is equal to some Q_i .

Assume first that all $P \in \mathcal{P}$ for which $P' = Q_i$ are equal to sxQ_i . Let $j \neq i$ and let $P \in \mathcal{P}$ such that $P' \in Q_j$. If $P = sxQ_j$, then P_1 is multicolored, where sx is colored by P and the remaining edges of P_1 are colored by paths from Q_i and P_1 . So, we may assume that for every $j \neq i$ and every $P \in \mathcal{P}$ for which $P' \in Q_j$ we have $P = sQ_j$. But this means that \mathcal{P} is regimented, the regimenting paths being sxQ_i and sQ_j , $j \neq i$.

Thus we may assume in this case that there exists $P \in \mathcal{P}$ such that $P' = Q_i$ but $P \neq sxQ_i$. There are two options: (i) $x \in V(P)$ and the initial path Px contains a vertex different from s and x . Let y be the last such vertex on Px . By Assertion 3.2 there exists $j \in \{2, \dots, m\}$ different from i , such that $y \in V(Q_j)$. But then the edge yx shows, by Assertion 3.5, that there exists a multicolored $s-t$ path. (ii) $x \notin V(P)$, that is $P = sQ_i$. Then P is a multicolored $s-t$, where the first edge is colored by P and the rest are colored by the paths in $Q_i \setminus \{P\}$ and P_1 .

Case II: The path xP_1 , with x replaced by s' , is not equal to any Q_i .

There are two subcases: (i) $V(xP_1 \setminus \{x\}) \subsetneq V(Q_i \setminus \{s'\})$. This means that there exist two consecutive vertices $u, v \in V(xP_1)$ such that there exist at least one vertex in $V(Q_i)$ between u and v . Then P_1 is a multicolored $s-t$, where uv is colored by P_1 and the rest are colored by the paths P such that $P' \in Q_i$. (ii) There exist two consecutive vertices u, v on P_1 such that the edge uv does not belong to $\bigcup_{2 \leq j \leq m} E(Q_j)$. By Assertion 3.2 this means that there exist $i \neq j$ in $\{2, \dots, m\}$ such that $u \in V(Q_i)$ and $v \in V(Q_j)$. Then, again by Assertion 3.5, there exists a multicolored $s-t$ path.

This concludes the proof of Theorem 3.3.

4. UNIQUENESS OF THE EXTREME CASE IN THEOREM 1.2

In this section we deduce Theorem 1.4 from Theorem 3.3. First, let us prove a weaker version of the theorem:

Lemma 4.1. *Under the conditions of Theorem 1.4, there exists a matching of size n representing at least $n-1$ matchings M_i .*

Proof. Add to the system M_1, \dots, M_{2n-2} a matching that is equal to one of the matchings M_i . By Theorem 1.2 there exists now a rainbow matching of size n , which may represent M_i twice, represents in all $n-1$ matchings. \square

Proof of Theorem 1.4

Let $F = \{e_1, \dots, e_{n-1}, e_n\}$ be a matching of size n , representing $n-1$ matchings among M_1, \dots, M_{2n-2} , as in Lemma 4.1. Without loss of generality we may assume that $e_i \in M_{n-1+i}$ for all $i < n$, and that $e_n \in M_n$. Thus, both e_1 and e_n represent M_n , each of M_{n+1}, \dots, M_{2n-2} is represented once, and M_1, \dots, M_{n-1} are not represented at all.

Let A, B be the sides of the bipartite graph, and let $e_i = a_i b_i$, where $a_i \in A$, $b_i \in B$. Assign to every edge e_i , $i < n$, a vertex v_i , and let $V = \{s, t, v_1, \dots, v_{n-1}\}$. An augmenting F -alternating path Q corresponds then to an $s-t$ path $P = P(Q)$ in a network \mathcal{N} on V , as follows. The first edge of Q , which is ub_i for some $u \in A \setminus \bigcup F$, is assigned the edge sv_i in $E(P)$. Every edge $a_i b_j \in E(Q)$, where $a_i, b_j \in \bigcup F$, is assigned the edge $v_i v_j \in E(P)$. The last edge of Q , which is $a_i w$ for some $w \in B \setminus \bigcup F$, is assigned the edge $v_i t$ in $E(P)$.

As in the proof of Theorem 1.2, each matching $M_i, i < n$ generates an F -alternating path Q_i , which then translates to an $s - t$ path $P_i = P(Q_i)$ in \mathcal{N} . By Theorem 3.3 we may assume that the system $\mathcal{P} = (P_1, \dots, P_{n-1})$ is regimented by a set $\mathcal{P}_j, j \in J$ of paths, where all paths in \mathcal{P}_j are equal to the same path P_j .

Assertion 4.2. *No two distinct paths in \mathcal{P} are connected by an edge belonging to some $E(Q_i), i < n$.*

Proof. Suppose that $\ell_1 \neq \ell_2$, and that $a_j b_k \in E(Q_i)$ for some $v_j \in V(P_{\ell_1}), v_k \in V(P_{\ell_2})$. Clearly, then, $i \notin \{\ell_1, \ell_2\}$, and $v_j \neq t, v_k \neq s$. These imply that $P_{\ell_1} v_j v_k P_{\ell_2}$ is an \mathcal{M} -multicolored $s - t$ path: there are enough paths in \mathcal{P}_{ℓ_1} to color $E(P_{\ell_1} v_j)$, enough paths in \mathcal{P}_{ℓ_2} to color $v_k P_{\ell_2}$, and the edge $v_j v_k$ is colored by P_i . \square

Assertion 4.3. *For every $i < n$ if $v_j \in V \setminus V(P_i)$ then $e_j \in M_i$.*

Proof. Let $E_i = \{v_j v_\ell \mid a_j b_\ell \in M_i\}$. Since $\bigcup V(P_i) = V$, and since E_i is the union of $V(P_i)$ and cycles, if $(v) \notin E_i$ then v lies on a non-singleton cycle C in E_i . By Assertion 4.2 $V(C) \subseteq V(P_j)$ for some $j \neq i$. Let ℓ be such that $P_j \in \mathcal{P}_\ell$. There exists an edge in C , say xy , that goes forward on P_j . Then the path $P_j xy P_j$ is \mathcal{M} -multicolored, where xy is colored by M_i and $E(P_j x) \cup E(y P_j)$ are colored by matchings M_k for which $P_k \in \mathcal{P}_\ell$. \square

Assertion 4.4. $|J| = 1$.

Proof. Suppose not. Then there exists $i < n$ such that $v_1 \notin V(P_i)$, and by Assertion 4.3 $e_1 \in M_i$. Re-coloring e_1 by M_i , and keeping the coloring of all other e_i 's (that is, e_i is colored by M_{n-1+i} for all $i \neq 1$) results then in a rainbow matching of size n . \square

Let the single path in the regimentation be $P = s v_{i_1} v_{i_2} \dots v_{i_{n-1}} t$. For each $k < n$ the fact that the first edge of $P, s v_{i_1}$, belongs to P_k means that there exists $c(k) \in A \setminus \{a_1, \dots, a_{n-1}\}$ such that $c(k) b_{i_1} \in P_k$.

Assertion 4.5. $c(k) = a_n$.

(Remember that $e_n = a_n b_n$ is one of the two edges of F belonging to M_n)

Proof. Assume that $c(k) \neq a_n$. Let j be such that $i_j = 1$. Apply to F the alternating path $P_k b_{i_j}$ (namely the initial part of P_k , up to and including b_{i_j}), and add the edge e_n . The resulting matching, $F \triangle E(P_k b_{i_j}) \cup \{e_n\}$, can be colored by n matchings M_i (e_n is colored by color 1), contradicting the negation assumption. \square

Symmetrically, the edge $v_{i_{n-1}} t$ represents the edge $a_{i_{n-1}} b_n$ in all P_k s. This means that all matchings $M_k, k \leq n-1$, are identical, each consisting of the edges $a_n b_{i_1}, a_{i_1} b_{i_2}, \dots, a_{i_{n-1}} b_n$.

In particular, this easily implies that there exists a matching of size n representing each of the matchings M_1, \dots, M_{n-1} , one of them twice. Applying a symmetric argument to the one above, we deduce that also M_n, \dots, M_{2n-2} are identical matchings, and this clearly implies that they are all equal to the matching F in the proof above. This shows that the matchings M_1, \dots, M_{2n-2} form a cycle, with the even edges being in the first n matchings and the odd edges being the last $n-1$ matchings, as desired.

5. THE EXTREME CASE IN THE ERDŐS-GINZBURG-ZIV THEOREM

The Erdős-Ginzburg-Ziv theorem [6] states that a multiset of $2n-1$ elements in \mathbb{Z}_n has a sub-multiset of size n summing up to 0 (mod n). Alon [2] noted that the EGZ theorem can be deduced from Theorem 1.2, as follows. Let A be a multi set of size $2n-1$ in \mathbb{Z}_n . For each $a \in A$ define a matching M_a in a bipartite graph with both sides indexed by \mathbb{Z}_n , by $M_a = \{(i, i+a) \mid i \in \mathbb{Z}_n\}$. By Theorem 1.2 there exists a sub-multiset B of size n of A , and a matching $(i, i+b(i)) \mid i \in \mathbb{Z}_n$, where $b(i) \in B$. Then the sum $\sum_{i \in \mathbb{Z}_n} i + b(i)$ includes in its terms every element of \mathbb{Z}_n precisely once, and hence $\sum_{i \in \mathbb{Z}_n} i + b(i) \equiv \sum_{i \in \mathbb{Z}_n} i \pmod{n}$, implying that $\sum_{i \in \mathbb{Z}_n} b(i) \equiv 0 \pmod{n}$.

An example showing tightness is a multiset consisting of $n-1$ copies of a and $n-1$ copies of b , where $\gcd(b-a, n) = 1$. The following was proved in [3, 7, 9], and here we give it yet another proof.

Theorem 5.1. *Any multiset of size $2n - 2$ in \mathbb{Z}_n not having a sub-multiset of size n summing up to $0 \pmod n$ is of the above form.*

Proof. Let A be a multiset with the above property. As shown above, if there is no multi-subset as desired then the set of matchings $M_c = \{(i, i + c) \mid i \in \mathbb{Z}_n\}$, $c \in A$, does not have a rainbow matching of size n . By Theorem 1.4 it follows that there are a, b such that $n - 1$ of the matchings M_c , $c \in A$, are equal to M_a , and $n - 1$ of them are equal to M_b . It remains to show that $\gcd(b - a, n) = 1$. If $\gcd(b - a, n) > 1$ then there exists k with $0 < k < n$ such that $k(b - a) \equiv 0 \pmod n$. Then the sum of k copies of b and $n - k$ copies of a is $0 \pmod n$, contrary to the assumption on A . \square

Acknowledgement We are indebted to Eli Berger for a helpful observation.

REFERENCES

- [1] R. Aharoni and E. Berger, rainbow matchings in r -partite hypergraphs, *Electronic J. of Comb.*, **16**, Issue 1 (2009).
- [2] N. Alon, Multicolored matchings in hypergraphs, *Moscow Journal of Combinatorics and Number Theory*, **1** (2011), 3–10.
- [3] N. Alon, A. Bialostocki and Y. Caro, The extremal cases in Erdős-Ginzburg-Ziv theorem, unpublished (1991)
- [4] A. A. Drisko, Transversals in row-Latin rectangles, *J. Combin. Theory Ser. A*, **84** (1998), 181–195.
- [5] J. Barát, A. Gyárfas and S. Sarkozy, Rainbow matchings in bipartite multigraphs, arXiv:1505.01779v2.
- [6] P. Erdős, A. Ginzburg and A. Ziv, A theorem in additive number theory, *Bull. Res. Council Israel* **10F**, (1961), 41–43.
- [7] C. Flores and O. Ordaz, On the Erdős-Ginzburg-Ziv theorem, *Discrete Mathematics*, **152** (1) (1996), 321–324.
- [8] S. K. Stein, Transversals of Latin squares and their generalizations, *Pacific J. Math.* **59** (1975), 567–575.
- [9] T. Yuster, Bounds for counter-example to addition theorem in solvable groups *Arch. Math. (Basel)*, **51** (1988), 223–231.

DEPARTMENT OF MATHEMATICS, TECHNION, HAIFA, ISRAEL

E-mail address, Ron Aharoni: raharoni@gmail.com

DEPARTMENT OF COMPUTER SCIENCE, TEL-HAI COLLEGE, ISRAEL

DEPARTMENT OF COMPUTER SCIENCE, TEL-HAI COLLEGE, ISRAEL